Kontsevich star-product on the dual of a Lie algebra

Giuseppe Dito
Laboratoire Gevrey de Mathématique physique
Université de Bourgogne, BP 47870, F-21078 Dijon Cedex, France
e-mail: ditog@u-bourgogne.fr

Dedicated to the memory of Moshé Flato.

Abstract

We show that on the dual of a Lie algebra $\mathfrak{g}$ of dimension d, the star-product recently introduced by M. Kontsevich is equivalent to the Gutt star-product on $\mathfrak{g}^*$. We give an explicit expression for the operator realizing the equivalence between these star-products.

1 Introduction

The study of formal (1-differentiable) deformations of the Lie-Poisson algebra of functions on a symplectic manifold was initiated in a paper by M. Flato, A. Lichnerowicz and D. Sternheimer [9]. Shortly afterwards, this program was extended to star-products, i.e., associative deformation of the usual product of functions, on a symplectic manifold [3] giving, among others, a profound interpretation of Quantum Mechanics as a deformation of Classical Mechanics in the direction of the Poisson bracket.

The existence problem of star-products has been solved by successive steps from special classes of symplectic manifolds to general Poisson manifolds. The existence of star-products on any finite dimensional symplectic manifold was first shown by M. De Wilde and P. Lecomte [5]. Since then, more geometric proofs have appeared [15, 8], and a proof of existence for regular Poisson manifolds was published by M. Masmoudi [16].

For the non-regular Poisson case, first examples of star-products appeared in [3] in relation with the quantization of angular momentum. They were defined on the dual of $\mathfrak{so}(n)$ endowed with its natural Kirillov-Poisson structure. The case for any Lie algebra follows from the construction given by S. Gutt [10] of a star-product on the cotangent bundle of a Lie group G. This star-product restricts to a star-product on the dual of the Lie algebra of G. It translates the associative structure of the enveloping algebra in terms of functions on the dual of the Lie algebra of G.

The problem of existence of star-products on any finite dimensional Poisson manifold was given a solution by M. Kontsevich [13]. The proof is based on

an explicit expression of a star-product on $\mathbb{R}^d$ endowed with a general Poisson bracket, which itself follows from more general formulae which allowed him to show his formality conjecture [12] for $\mathbb{R}^d$ and then for any finite dimensional manifold M. Recently, by using different techniques, D. Tamarkin [19] has indicated another proof of the formality conjecture. This is one of the ingredients in the most recent fundamental paper by M. Kontsevich [14].

On the dual of a Lie algebra, we have a priori two different star-products: Gutt and Kontsevich star-products. D. Arnal [1] showed that when the Lie algebra is nilpotent, these two star-products do coincide. Here we shall give an elementary proof that in the general case Gutt and Kontsevich star-products are equivalent and explicitly construct the equivalence between them. For that purpose, we use the notion of Weyl star-products on $\mathbb{R}^d$. These are star-products enjoying the following property: $X *_\hbar \cdots *_\hbar X$ (k factors) is equal to X^k (usual product) for any linear polynomial Xon $\mathbb{R}^d$ and $k \geq 0$. Any star-product on $\mathbb{R}^d$ is equivalent to a Weyl star-product and, in the case of the dual of a Lie algebra, Gutt star-product is the unique covariant Weyl star-product. From this fact, one immediately obtains that Kontsevich and Gutt star-products are equivalent. The equivalence operator is obtained by applying a method used in [7] in the context of generalized Abelian deformations. It turns out that the equivalence operator is an exponential of constant coefficients linear operators given by the trace of powers of the adjoint map of the Lie algebra. The formula we have found is closely related to the discussion given in [13] about Lie algebras.

The paper is organized as follows. We review Kontsevich construction in Section 2. Then we proceed to the study of Weyl star-products on $\mathbb{R}^d$ and get a characterization of Gutt star-product. The main results about equivalence is proved in Section 4. We end the paper with some remarks on the general Poisson case.

Since the first version of this paper was completed, several preprints dealing with Kontsevich star-product on the dual of a Lie algebra have appeared [2, 11, 17]. The equivalence result found here has also been obtained by D. Arnal, N. Ben Amar, and M. Masmoudi [2] in a completely different approach involving cohomology of Kontsevich graphs.

2 Kontsevich star-product

The reader is referred to [3] for the general theory on star-products and to the excellent review by D. Sternheimer [18] for further details and recent applications.

We shall briefly review the construction of a star-product on $\mathbb{R}^d$ given in [13]. Consider $\mathbb{R}^d$ endowed with a Poisson bracket π. We denote by $(x^1, \ldots, x^d)$ the coordinate system on $\mathbb{R}^d$, the Poisson bracket of two smooth functions f, g is given by $\pi(f,g) = \sum_{1 \leq i,j \leq n} \pi^{ij} \partial_i f \partial_j g$, where ∂_k denotes the partial derivative with respect to x^k. What follows remain valid if, instead of the whole of $\mathbb{R}^d$, one considers an open subset of it. We slightly depart from the notations used in [13].

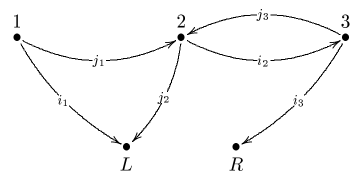


Figure 1: A typical graph in G_3.

The formula for Kontsevich star-product is conveniently defined by considering, for each $n \geq 0$, a family of oriented graphs G_n. To a graph $\Gamma \in G_n$ is associated a bidifferential operator $\mathcal{B}_\Gamma$ and a weight $w(\Gamma) \in \mathbb{R}$. The sum $\sum_{\Gamma \in G_n} w(\Gamma)\mathcal{B}_\Gamma$ gives us the term at order $\hbar^n$, i.e., the cochain C_n of the star-product. Here is the formal definition of G_n.

An oriented graph Γ belongs to G_n, $n \geq 0$, if:

i) Γ has $n+2$ vertices labeled $\{1, 2, \ldots, n, L, R\}$ where L and R stand for Left and Right, respectively, and Γ has $2n$ oriented edges labeled $\{i_1, j_1, i_2, j_2, \ldots, i_n, j_n\}$;

ii) The pair of edges $\{i_k, j_k\}$, $1 \leq k \leq n$, starts at vertex k;

iii) Γ has no loop (edge starting at some vertex and ending at that vertex) and no parallel multiple edges (edges sharing the same starting and ending vertices).

When it is needed to make explicit at which vertex $v \in \{1, \ldots, n, L, R\}$ some edge, e.g. j_k, is ending at, we shall use the notation $j_k(v)$.

The set of graphs in G_n is finite. For $n \geq 1$, the first edge i_k starting at vertex k has $n+1$ possible ending vertices (since there is no loop), while the second edge j_k has only n possible ending vertices, since there is no parallel multiple edges. Thus there are $n(n+1)$ ways to draw the pair of edges starting at some vertex and therefore G_n has $(n(n+1))^n$ elements. For $n = 0$, G_0 has only one element: The graph having as set of vertices $\{L, R\}$ and no edges.

A bidifferential operator $(f, g) \mapsto \mathcal{B}_\Gamma(f, g)$, $f, g \in C^\infty(\mathbb{R}^d)$, is associated to each graph $\Gamma \in G_n$, $n \geq 1$. To each vertex k, $1 \leq k \leq n$, one associates the components $\pi^{i_k j_k}$ of the Poisson tensor, f is associated to the vertex L and g to the vertex R. Each edge, e.g. $i_k(v)$ acts by partial differentiation with respect to x^{i_k} on its ending vertex v. There is no better way than to draw the graph Γ to illustrate the correspondence $\Gamma \mapsto \mathcal{B}_\Gamma$. See [13] for a general formula. The graph in Fig. 1 gives the bidifferential operator

$$\mathcal{B}_\Gamma(f, g) = \sum_{0 \leq i_*, j_* \leq d} \pi^{i_1 j_1}\, \partial_{j_1 j_3} \pi^{i_2 j_2}\, \partial_{i_2} \pi^{i_3 j_3}\, \partial_{i_1 j_2} f\, \partial_{i_3} g.$$

Notice that for $n = 0$, we simply have the usual product of f and g.

Now let us describe how the weight $w(\Gamma)$ of a graph Γ is defined. Again the reader is referred to [13] for details and a nice geometrical interpretation of

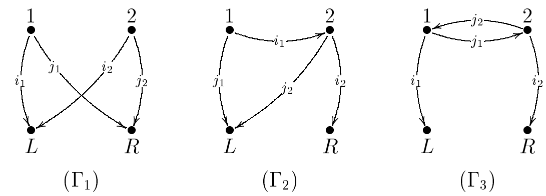


Figure 2: Graphs contributing to C_2^K.

what follows. Let $\mathcal{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ be the upper half-plane. $\mathcal{H}_n$ will denote the configuration space $\{z_1, \ldots, z_n \in \mathcal{H} \,|\, z_i \neq z_j \text{ for } i \neq j\}$. $\mathcal{H}_n$ is an open submanifold of $\mathbb{C}^n$. Let $\phi\colon \mathcal{H}_2 \to \mathbb{R}/2\pi\mathbb{Z}$ be the function:

$$\phi(z_1, z_2) = \frac{1}{2\sqrt{-1}} \mathrm{Log}\Big(\frac{(z_2 - z_1)(\bar{z}_2 - z_1)}{(z_2 - \bar{z}_1)(\bar{z}_2 - \bar{z}_1)}\Big). \tag{1}$$

$\phi(z_1, z_2)$ is extended by continuity for $z_1, z_2 \in \mathbb{R}$, $z_1 \neq z_2$.

For a graph $\Gamma \in G_n$, the vertex k, $1 \leq k \leq n$, is associated with the variable $z_k \in \mathcal{H}$, the vertex L with $0 \in \mathbb{R}$, and the vertex R with $1 \in \mathbb{R}$.

The weight $w(\Gamma)$ is defined by integrating an $2n$-form over $\mathcal{H}_n$:

$$w(\Gamma) = \frac{1}{n!(2\pi)^{2n}} \int_{\mathcal{H}_n} \bigwedge_{1 \leq k \leq n} \Big(d\phi(z_k, I_k) \wedge d\phi(z_k, J_k)\Big), \tag{2}$$

where I_k (resp. J_k) denotes the variable or real number associated with the ending vertex of the edge i_k (resp. j_k). For example, the weight of the graph in Fig. 1 consists in integrating the 6-form $d\phi(z_1, 0) \wedge d\phi(z_1, z_2) \wedge d\phi(z_2, z_3) \wedge d\phi(z_2, 0) \wedge d\phi(z_3, 1) \wedge d\phi(z_3, z_2)$ on $\mathcal{H}_3$. It is clear from the definition of the weights that they are universal in the sense that they do not depend on the Poisson structure or the dimension d.

The origin of the weights has been elucidated by A. S. Cattaneo and G. Felder [4]. These authors have been able to construct a bosonic topological field theory on a disc whose perturbation series (after a finite renormalization taking care of tadpoles) makes Kontsevich graphs and weights appear explicitly.

It is showed in [13] that the integral in Eq. (2) is absolutely convergent. A pillar result in [13] is

Theorem 1 (Kontsevich) *For any Poisson structure π on $\mathbb{R}^d$, the map*

$$(f, g) \mapsto \sum_{n \geq 0} \hbar^n \sum_{\Gamma \in G_n} w(\Gamma) \mathcal{B}_\Gamma(f, g), \quad f, g \in C^\infty(\mathbb{R}^d),$$

defines an associative product.

We call this product the Kontsevich star-product and it will be denoted by $*_\hbar^K$ and the corresponding cochains by C_r^K. Actually the preceding theorem holds if π is replaced by any formal Poisson bracket $\pi_\hbar = \pi + \sum_{r \geq 1} \hbar^r \pi_r$.

Moreover, equivalence classes of star-products are in one-to-one correspondence with equivalence classes of formal Poisson brackets [13].

At first it may seem that any computation involving the graphs becomes rapidly cumbersome as $\#G_n = (n(n+1))^n$. But the situation is not that bad, there are many isomorphic graphs obtained by permuting the vertices or interchanging the edges $\{i_k, j_k\} \to \{j_k, i_k\}$. These operations do not affect $w(\Gamma)\mathcal{B}_\Gamma$ as each factor picks up a minus sign. Also in G_n, $n \geq 2$, there are "bad" graphs that can be eliminated right away. These graphs are those for which the vertices L or R (or both) do not receive any edge. As it should, the weights associated to these graphs vanish.

We will illustrate that by giving the explicit form of the second cochain C_2 which requires at the end the computation of only 3 graphs (notice that $\#G_2 = 36$).

The graphs in Fig. 2 have weights $w(\Gamma_1) = 1/8$, $w(\Gamma_2) = 1/24$, $w(\Gamma_3) = -1/48$. By counting the symmetries, the graph Γ_1 contributes 4 times, Γ_2 contributes 8 times, and Γ_3 contributes 8 times. There is also a sister-graph for Γ_2 which is obtained by performing $L \leftrightarrow R$ which contributes also 8 times. Taking into account that there are 8 "bad" graphs, we a have a total of 36 graphs, and we find that:

$$
\begin{aligned}
C_2^K(f,g) \quad = \quad & \frac{1}{2}\,\pi^{i_1 j_1}\,\pi^{i_2 j_2}\,\partial_{i_1 i_2} f\,\partial_{j_1 j_2} g \\
& + \frac{1}{3}\,\pi^{i_1 j_1}\,\partial_{i_1}\pi^{i_2 j_2}\,(\partial_{j_1 j_2} f\,\partial_{i_2} g + \partial_{i_2} f\,\partial_{j_1 j_2} g) \\
& - \frac{1}{6}\,\partial_{j_2}\pi^{i_1 j_1}\,\partial_{j_1}\pi^{i_2 j_2}\,\partial_{i_1} f\,\partial_{i_2} g,
\end{aligned}
\tag{3}
$$

where summation over repeated indices is understood.

3 Weyl star-products on $\mathbb{R}^d$

Let π be a general Poisson structure on $\mathbb{R}^d$. Let Pol be the algebra of polynomials in the variables $x^1, \ldots, x^d$ and let Lin denote the subspace of linear homogeneous polynomials. Let $*_\hbar$ be a star-product on $(\mathbb{R}^d, \pi)$. We shall show that $*_\hbar$ is (differentially) equivalent to a star-product ${*_\hbar}'$ having the following property:

$$
X^{{*_\hbar}' k} = X^k, \quad \forall k \geq 0, \forall X \in \mathsf{Lin}, \tag{4}
$$

where $X^{{*_\hbar}' k} = X {*_\hbar}' \cdots {*_\hbar}' X$ (k factors). This is reminiscent of Weyl ordering in Quantum Mechanics and we introduce:

Definition 1 *A star-product on $(\mathbb{R}^d, \pi)$ satisfying Eq. (4) is called a Weyl star-product.*

The consideration of this kind of star-products amount to generalized Abelian deformations [6, 7]. We recall the proof of the following:

Theorem 2 *Any star-product on $(\mathbb{R}^d, \pi)$ is equivalent to a Weyl star-product.*

Proof. The proof consists to establish the differentiability of the following $\mathbb{R}[[\hbar]]$-linear map $\rho\colon \mathsf{Pol}[[\hbar]] \to C^\infty(\mathbb{R}^d)[[\hbar]]$ uniquely defined by:

$$\rho(X^k) = X^{*_\hbar k}, \quad \forall k \geq 0, \forall X \in \mathsf{Lin}. \tag{5}$$

The map ρ is a formal sum of linear maps $\rho = \sum_{i\geq 0} \hbar^i \rho_i$ with ρ_0 being the identity map on Pol. We will show that the ρ_r's are differential operators. By definition $\rho_r(1) = \rho_r(X) = 0$ for $r \geq 1$ and $X \in \mathsf{Lin}$. It is easy to see from Eq. (5) that the ρ_r's satisfy the following recurrence relation for $k \geq 1, r \geq 1$:

$$-\delta\rho_r(X, X^{k-1}) = C_r(X, X^{k-1}) + \sum_{\substack{a+b=r \\ a,b\geq 1}} C_a(X, \rho_b(X^{k-1})), \tag{6}$$

(the C_r's are the 2-cochains of the star-product). For $r = 1$ the sum on the right-hand side is omitted and δ is the Hochschild differential. Before going further we need a lemma.

Lemma 1 *Let $\psi\colon \mathsf{Pol} \to C^\infty(\mathbb{R}^d)$ be an $\mathbb{R}$-linear map such that $\psi(1) = \psi(X) = 0$, for $X \in \mathsf{Lin}$, and let $\phi\colon C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ be a bidifferential operator vanishing on constants. If ψ satisfies*

$$\delta\psi(X, X^{k-1}) = \phi(X, X^{k-1}), \quad \forall k \geq 1, \forall X \in \mathsf{Lin}, \tag{7}$$

then there exists a differential operator η on $\mathbb{R}^d$ such that $\psi = \eta|_{\mathsf{Pol}}$.

Proof. For two functions f, g, let $\sum_{I,J} \phi^{I,J} \partial_I f \partial_J g$ be the expression of $\phi(f, g)$ in local coordinates, where I and J are multi-indices and $\phi^{I,J}$ is a smooth function vanishing for $|I|$ or $|J|$ greater than some integer (for $I = (i_1, \ldots, i_n)$, $|I|$ denotes its length $i_1 + \cdots + i_n$). In Eq. (7), only first derivatives can be applied to the first argument of ϕ and one can check the following series of equalities:

$$\phi(X, X^{k-1}) = \sum_{i,J} \phi^{i,J} \partial_i X \partial_J X^{k-1} = \frac{1}{k} \sum_{i,J} \phi^{i,J} \partial_{iJ} X^k = \delta\eta(X, X^{k-1}), \tag{8}$$

where $\eta = -\sum_{i,J} \frac{1}{1+|J|} \phi^{i,J} \partial_{iJ}$. Now Eq. (7) can be written as $\delta(\psi - \eta)(X, X^{k-1}) = 0$ and by observing that $\psi - \eta$ vanishes on 1 and X we get that $\psi - \eta = 0$ on Pol. This shows the lemma.

The term of order 1 in Eq. (6) yields $(\delta\rho_1 + C_1)(X, X^{k-1}) = 0$. We have that $C_1 = \pi + \delta\theta_1$ for some differentiable 1-cochain θ_1, which can be chosen such that $\theta_1(X) = 0$ for $X \in \mathsf{Lin}$ by adding an appropriate Hochschild 1-cocycle (i.e., a vector field). Then as before $\delta(\rho_1 + \theta_1)(X, X^{k-1}) = 0$ gives us $\rho_1 = -\theta_1$ on Pol showing that ρ_1 is a differential operator.

With the help of Lemma 1, a simple recurrence on r in Eq. (6) shows that for each $r \geq 1$, ρ_r coincides with the restriction of a differential operator to Pol. Clearly the map ρ can be naturally extended to an $\mathbb{R}[[\hbar]]$-linear map on $C^\infty(\mathbb{R}^d)[[\hbar]]$. We still denote this extension by ρ.

The map ρ is invertible as ρ_0 is the identity map and we can use it to define an equivalent star-product ${*_\hbar}'$ to $*_\hbar$ by:

$$\rho(f{*_\hbar}'g) = \rho(f){*_\hbar}\rho(g), \quad f,g \in C^\infty(\mathbb{R}^d). \tag{9}$$

Notice that $X^{{*_\hbar}'k} = \rho^{-1}(\rho(X)^{*_\hbar k}) = \rho^{-1}(X^{*_\hbar k}) = X^k$ for $\forall k \geq 0$ and $\forall X \in \mathsf{Lin}$, therefore ${*_\hbar}'$ is a Weyl star-product.

3.1 Gutt star-product on $\mathfrak{g}^*$

Let G be a real finite-dimensional group of dimension d. The Lie algebra of G is denoted by $\mathfrak{g}$ and its dual by $\mathfrak{g}^*$. The universal enveloping algebra (resp. symmetric algebra) of $\mathfrak{g}$ is denoted by $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{S}(\mathfrak{g})$). Also we denote by $\mathsf{Pol}(\mathfrak{g}^*)$ the space of polynomials on $\mathfrak{g}^*$.

It is well known that the space of smooth functions on $\mathfrak{g}^*$ carries a natural Poisson structure defined by the Kirillov-Poisson bracket denoted by Π. Fix a basis for $\mathfrak{g}$, let C_{ij}^k be the structure constants in that basis, and let $(x^1, \ldots, x^d)$ be the coordinates on $\mathfrak{g}^*$. Then the Kirillov-Poisson bracket is defined by:

$$\Pi(f,g) = \sum_{1 \leq i,j,k \leq d} x^k\, C_{ij}^k\, \partial_i\, f \partial_j g, \quad f,g \in C^\infty(\mathfrak{g}^*). \tag{10}$$

Of course this definition is independent of the chosen basis for $\mathfrak{g}$.

S. Gutt has defined a star-product on the cotangent bundle T^*G of a Lie group G [10]. When one restricts this star-product between functions not depending on the base point in G, one gets a star-product on $\mathfrak{g}^*$. We shall call the induced product on $C^\infty(\mathfrak{g}^*)$, Gutt star-product on $\mathfrak{g}^*$ and denote it by $*_\hbar^G$. Gutt star-product on $\mathfrak{g}^*$ can also be directly obtained by transporting the algebraic structure of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ to the space of polynomials on $\mathfrak{g}^*$. This is achieved through the natural isomorphism between $\mathsf{Pol}(\mathfrak{g}^*)$ and $\mathcal{S}(\mathfrak{g})$ and with the help of the symmetrization map $\sigma\colon \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$. The product between two homogeneous elements P and Q in $\mathcal{S}(\mathfrak{g}) \sim \mathsf{Pol}(\mathfrak{g}^*)$ of degrees p and q, respectively, is given by:

$$P *_\hbar^G Q = \sum_{0 \leq r \leq p+q-1} (2\hbar)^r \sigma^{-1}((\sigma(P)\cdot\sigma(Q))_{p+q-r}), \tag{11}$$

where $\cdot$ is the product in $\mathcal{U}(\mathfrak{g})$ and, for $v \in \mathcal{U}(\mathfrak{g})$, $(v)_k$ means the k-th component of v with respect to the associated grading of $\mathcal{U}(\mathfrak{g})$. Formula (11) defines an associative deformation of the usual product on $\mathsf{Pol}(\mathfrak{g}^*)$ which admits a unique extension to $C^\infty(\mathfrak{g}^*)$.

As a direct consequence of Eq. (11) we have that $*_\hbar^G$ is a Weyl star-product on $(\mathfrak{g}^*, \Pi)$. Moreover the following property is easily verified:

$$X *_\hbar Y - Y *_\hbar X = 2\hbar\Pi(X,Y), \quad X, Y \in \mathsf{Lin}(\mathfrak{g}^*),$$

where $\mathsf{Lin}(\mathfrak{g}^*)$ is the subspace of homogeneous polynomials of degree 1 on $\mathfrak{g}^*$. Star-products on $(\mathfrak{g}^*, \Pi)$ satisfying the preceding relation are called $\mathfrak{g}$-covariant star-products. Actually, there is a characterization of Gutt star-product given by:

Lemma 2 *Gutt star-product is the unique $\mathfrak{g}$-covariant Weyl star-product on $(\mathfrak{g}^*, \Pi)$. Any $\mathfrak{g}$-covariant star-product on $(\mathfrak{g}^*, \Pi)$ is equivalent to Gutt star-product.*

Proof. Any star-product $*_\hbar$ on $(\mathfrak{g}^*, \Pi)$ is determined by the quantities $\exp(X) * \exp(Y)$, $X, Y \in \mathsf{Lin}(\mathfrak{g}^*)$. Suppose that $*_\hbar$ is a $\mathfrak{g}$-covariant Weyl star-product, then the star-exponential of $X \in \mathsf{Lin}(\mathfrak{g}^*)$ defined by:

$$\exp_{*_\hbar}(X) = \sum_{k \geq 0} \frac{1}{k!} X^{*_\hbar k},$$

coincides with the usual exponential $\exp(X)$. The covariance property of $*_\hbar$ allows to use the Campbell-Hausdorff formula:

$$\exp(X) *_\hbar \exp(Y) = \exp_{*_\hbar}(CH_\hbar(X,Y)) = \exp(CH_\hbar(X,Y)), \tag{12}$$

where $CH_\hbar(X,Y)$ is the usual Campbell-Hausdorff series with respect to the bracket $[X,Y] = 2\hbar\Pi(X,Y)$. For $X, Y \in \mathsf{Lin}(\mathfrak{g}^*)$, $CH_\hbar(X,Y)$ is an element of $\mathsf{Lin}(\mathfrak{g}^*)[[\hbar]]$. We still have $CH_\hbar(X,Y)^{*_\hbar k} = CH_\hbar(X,Y)^k$ for $k \geq 0$. It follows from Eq. (12) that there is at most one $\mathfrak{g}$-covariant Weyl star-product on $\mathfrak{g}^*$, i.e., Gutt star-product. The second statement of the lemma follows from Theorem 2 and from the fact that the equivalence operator ρ preserves the covariance property, i.e., $\rho(X) = X$ for $X \in \mathsf{Lin}(\mathfrak{g}^*)$ (cf. Eq. (5)).

From Eq. (12), one can derive an explicit expression for the cochains of Gutt star-product. Denote by c_i, $i \geq 1$, the Campbell-Hausdorff coefficients: $c_1(X,Y) = X+Y$, $c_2(X,Y) = \frac{1}{2}[X,Y]$, etc. The term of order $\hbar^r$ in $\exp(X) *^G_\hbar \exp(Y)$ for $X, Y \in \mathsf{Lin}(\mathfrak{g}^*)$ is obtained by expanding in powers of $\hbar$ the right hand-side in Eq. (12), it is given by:

$$\begin{aligned} & C^G_r(\exp(X), \exp(Y)) \\ & \quad = 2^r \exp(X+Y) \sum_{1 \leq k \leq r} \sum_{\substack{m_1 > \cdots > m_k \geq 1 \\ n_1, \ldots n_k \geq 1 \\ m_1 n_1 + \cdots + m_k n_k = r}} \prod_{1 \leq j \leq k} \frac{1}{n_j!} (c_{m_j+1}(X,Y))^{n_j}, \end{aligned} \tag{13}$$

where the bracket $[X,Y]$ in the c_i's is taken to be $\Pi(X,Y)$. For $r = 2$, we easily get the differential expression for $C^G_2(f,g)$, $f, g \in C^\infty(\mathfrak{g}^*)$:

$$\frac{1}{2!} \Pi^{i_1 j_1} \Pi^{i_2 j_2} \partial_{i_1 i_2} f \, \partial_{j_1 j_2} g + \frac{1}{3} \Pi^{i_1 j_1} \partial_{i_1} \Pi^{i_2 j_2} (\partial_{j_1 j_2} f \, \partial_{i_2} g + \partial_{i_2} f \, \partial_{j_1 j_2} g).$$

Comparing with the general expression of the second cochain of Kontsevich star-product given by (3), we see that in general Kontsevich and Gutt star-products differ. Notice that the extra term in (3) is a Hochschild 2-coboundary.

It is instructive to derive from Eq. (13) an expression for $X *^G_\hbar g$, $X \in \mathsf{Lin}(\mathfrak{g}^*)$, $g \in C^\infty(\mathfrak{g}^*)$. Using the standard recurrence formula for the c_i's, it is easy to establish that

$$c_i(0,X) = c_i(X,0) = 0, \quad i \geq 2;$$

$$\frac{\partial}{\partial s} c_i(sX, Y)|_{s=0} = \frac{B_{i-1}}{(i-1)!}(ad_Y)^{i-1}(X), \quad i \geq 2; \tag{14}$$

for $X \in \mathsf{Lin}(\mathfrak{g}^*)$, $s \in \mathbb{R}$, ad_Y is the adjoint map $X \mapsto [Y, X]$, and the B_n's are the Bernoulli numbers. The substitution $X \to sX$ in Eq. (13) and differentiation with respect to s gives:

$$C_r^G(X, \exp(Y)) = \frac{2^r B_r}{r!}(ad_Y)^r(X)\ \exp(Y),$$

which leads to

$$C_r^G(X, g) = (-1)^r \frac{2^r B_r}{r!} \sum_{1 \leq i_*, j_* \leq d} \Pi^{i_1 j_1} \partial_{i_1} \Pi^{i_2 j_2} \cdots \partial_{i_{r-1}} \Pi^{i_r j_r} \partial_{i_r} X \partial_{j_1 \cdots j_r} g. \tag{15}$$

Remark 1 *From Eq. (13) or, better, Eq. (15), it is clear that the weights of the graphs appearing in Gutt star-product are essentially products of Bernoulli numbers.*

4 Equivalence

In this section, as in Sect. 3, we consider a Lie algebra $\mathfrak{g}$ of dimension d and use the notations previously introduced. We have seen that in general Kontsevich and Gutt star-products are not identical. We will show that they are equivalent and explicitly determine the equivalence operator by computing a subfamily of graphs.

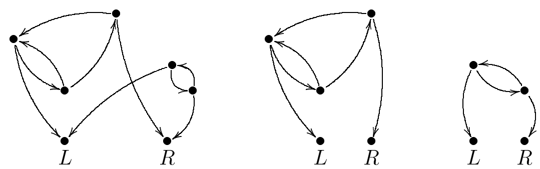
Lemma 3 *Kontsevich star-product $*_\hbar^K$ on $(\mathfrak{g}^*, \Pi)$ is a $\mathfrak{g}$-covariant star-product.*

Proof. We just need to see what kind of graphs contribute to $X *_\hbar^K Y$, $X, Y \in \mathsf{Lin}(\mathfrak{g}^*)$. The graphs for $C_r^K(X, Y)$ must be such that the vertices L and R receive only one edge, respectively. For $r = 1$, we simply have the Poisson bracket Π. If $r \geq 2$, we need to draw $2r - 2$ edges in such a way that each vertex k, $1 \leq k \leq r$ receives at most one edge (since the Poisson bracket Π is linear in the coordinates) and this is possible only if $2r - 2 \leq r$, i.e., $r \leq 2$. For $r = 2$, the only graph contributing (up to symmetry factors) is the graph Γ_3 in Fig. 2, whose associated bidifferential operator $\mathcal{B}_{\Gamma_3}$ is symmetric. Thus we have $X *_\hbar Y - Y *_\hbar X = 2\hbar\Pi(X, Y)$.

As a consequence of Lemmas 2 and 3 we have:

Corollary 1 *On the dual of a Lie algebra, Kontsevich and Gutt star-products are equivalent.*

The formal series of differential operators realizing the equivalence between Kontsevich and Gutt star-products is the map ρ defined in the proof of Theorem 2. We have $\rho(f *_\hbar^G g) = \rho(f) *_\hbar^K \rho(g)$, $\forall f, g \in C^\infty(\mathfrak{g}^*)$ and in the present



(Γ) (Γ_1) (Γ_2)

Figure 3: Union of graphs: $\Gamma = \Gamma_1 \cup \Gamma_2$.

situation ρ is defined by $\rho(X^k) = X^{*^K_\hbar k}$, $X \in \mathsf{Lin}(\mathfrak{g}^*)$, $k \geq 0$. We will see (cf. Theorem 3) that to solve the recurrence relation (6) satisfied by the ρ_r's, it is sufficient to consider graphs contributing to $C^K_r(X, X^k)$. The graphs having a non-trivial contribution must have only one edge ending at vertex L, e.g. i_k, and the other edge j_k must end at some vertex $k' \neq k$, $1 \leq k' \leq r$.

We shall say that a graph $\Gamma \in G_r$ is the union of two subgraphs $\Gamma_1 \in G_{r_1}$ and $\Gamma_2 \in G_{r_2}$ with $r_1 + r_2 = r$, if the subset $(1, \ldots, r)$ of the set of vertices of Γ can be split into two parts $(a_1, \ldots, a_{r_1})$ and $(b_1, \ldots, b_{r_2})$ such that there is no edge between these two subsets of vertices. A graph that is not the union of two subgraphs is called indecomposable.

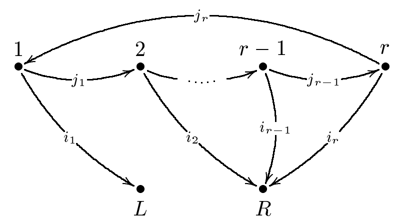
By recalling the definition of the weight of a graph, the following is straightforward:

Lemma 4 *If a graph $\Gamma \in G_r$ is the union of two subgraphs Γ_1 and Γ_2, respectively, in G_{r_1} and in G_{r_2} with $r_1 + r_2 = r$, then $w(\Gamma) = \frac{r_1! r_2!}{r!} w(\Gamma_1) w(\Gamma_2)$.*

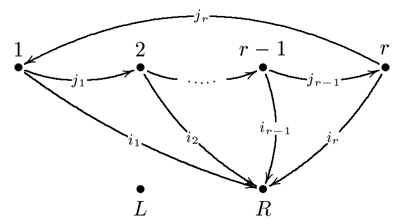
In view of this lemma, we just need to confine ourselves to indecomposable graphs whose union is contributing to $C^K_r(X, X^k)$.

Lemma 5 *For $r \geq 2$, up to an isomorphism of graphs, an indecomposable graph in G_r contributing to $C^K_r(X, X^k)$ falls into one of the two types illustrated in Fig. 4.*

Proof. As the vertex L can receive at most one edge, we distinguish two cases.
i) The vertex L receives no edge. We will see that the vertex R must receive exactly r edges. If there are strictly more than r edges ending at vertex R, then there must be a vertex k, $1 \leq k \leq r$, such that the edges i_k and j_k are ending at vertex R. This is excluded by definition of G_r. If there are strictly less than r edges ending at vertex R, then at least one of the vertices $(1, \ldots, r)$ must receive two or more edges and the bidifferential operator associated to such a graph is vanishing since the Poisson bracket is a linear function of the coordinates. We are left with the case where exactly r edges are ending at vertex R. Then every vertex in $(1, \ldots, r)$ must receive exactly one edge and, up to an isomorphism, there is precisely one such a graph, i.e., graph $\Gamma_2^{(r)}$ in Fig. 4.
ii) The vertex L receives one edge. For this case, the vertex R receives $r-1$ edges. By relabeling the vertices and the edges, we may suppose that the edge ending at vertex L is i_1. Then the second edge starting at the vertex 1, i.e.,



$(\Gamma_1^{(r)})$



$(\Gamma_2^{(r)})$

Figure 4: Indecomposable graphs.

j_1, cannot end at vertex R because, by skew-symmetry of the Poisson bracket, the associated bidifferential operator is vanishing on (X, X^k). Hence we may suppose that the edge j_1 is ending at vertex 2. We still have $2r-2$ edges starting from the vertices $(2, \ldots, r)$ to draw. Let a be the number of edges ending at vertex R and let b be the total number of edges ending at the vertices $(1, \ldots, r)$. We have $2r - 1 = a + b$. Since each vertex in $(1, \ldots, r)$ can receive at most one edge, we have that $b \leq r$ and it follows that $a \geq r-1$. If $a > r-1$, it means that there are parallel multiple edges between at least one of the vertices $(2, \ldots, r)$ and the vertex R. Hence the vertex R must receive $r - 1$ edges. Clearly every such edge must start at one the vertices $(2, \ldots, r)$. The other $r - 1$ edges must end at the vertices $(1, 3, \ldots, r)$. Thus, up to an isomorphism, we find that there is only the graph $\Gamma_1^{(r)}$ in Fig. 4 for this case.

The preceding lemma tells us that graphs contributing to $C_r^K(X, X^k)$ must be of the form:

$$\Gamma_1^{(a)} \cup \Gamma_2^{(b_1)} \cup \cdots \cup \Gamma_2^{(b_s)},$$

with $a + b_1 + \cdots + b_s = r$. Notice that there can be only one graph of the type $\Gamma_1^{(a)}$, since the vertex L can receive only one edge. Quite a bit of simplification is allowed by

Lemma 6 *For $r \geq 2$, the weight of the graph $\Gamma_2^{(r)}$ in Fig. 4 vanishes.*

Proof. The form $\bigwedge_{1 \leq k \leq r} d\phi(z_k, 1) \wedge d\phi(z_k, z_{k+1})$, where $z_{r+1} \equiv z_1$, is 0. This easily follows from a simple recurrence using explicit expressions for the forms

$d\phi(z_i, z_j)$.

When they appear alone, the graphs $\Gamma_2^{(r)}$ constitute an example of what was called "bad" graphs in Sect. 2.

Lemma 7 *For $r \geq 2$, up to an isomorphism, the only graph contributing to $C_r^K(X, X^k)$ is the graph $\Gamma_1^{(r)}$ in Fig. 4. The associated bidifferential operator has constant coefficients and is given by:*

$$\mathcal{B}_{\Gamma_1^{(r)}}(f,g) = \sum_{1 \leq i_* \leq d} \mathsf{Tr}(ad_{x^{i_1}} \cdots ad_{x^{i_r}})\, \partial_{i_1} f\, \partial_{i_2 \cdots i_r} g, \quad r \geq 2. \tag{16}$$

Proof. The first statement follows directly from Lemmas 4, 5, and 6. The bidifferential operator for the graph $\Gamma_1^{(r)}$ is

$$\mathcal{B}_{\Gamma_1^{(r)}}(f,g) = \sum_{1 \leq i_*, j_* \leq d} \partial_{j_r} \Pi^{i_1 j_1} \partial_{j_1} \Pi^{i_2 j_2} \cdots \partial_{j_{r-1}} \Pi^{i_r j_r}\, \partial_{i_1} f\, \partial_{i_2 \cdots i_r} g,$$

clearly it has constant coefficients and using the expression (10) for Π, we see that the previous equation can be written as a trace of adjoint maps.

The computation of the weights of the graphs $\Gamma_1^{(r)}$ is a delicate question. The presence of cycles (wheels) does not allow do derive a simple recurrence relation among the weights. A direct calculation for $\Gamma_1^{(2)}$ using residues gives a weight equals to $-1/48$, but this method becomes unpractical for $r \geq 3$.

The isomorphic graphs to $\Gamma_1^{(r)}$ are obtained by permuting the vertices $(1, \ldots, r)$ and alternating the edges $\{i_k, j_k\} \to \{j_k, i_k\}$ for $1 \leq k \leq r$, thus we get a symmetry factor $r!2^r$. Hence we have $C_r^K(X, X^k) = 2^r r! w(\Gamma_1^{(r)}) \mathcal{B}_{\Gamma_1^{(r)}}(X, X^k)$.

Theorem 3 *On any finite-dimensional Lie algebra, the equivalence ρ between Kontsevich and Gutt star-products: $\rho(f *_\hbar^G g) = \rho(f) *_\hbar^K \rho(g)$, is given by*

$$\rho = \exp\Big(\sum_{r \geq 2} \hbar^r 2^r (r-1)! w(\Gamma_1^{(r)}) D_r\Big), \tag{17}$$

where the D_r's are differential operators with constant coefficients:

$$D_r = \sum_{1 \leq i_* \leq d} \mathsf{Tr}(ad_{x^{i_1}} \cdots ad_{x^{i_r}})\, \partial_{i_1 \cdots i_r}.$$

Proof. Recall that ρ is defined by $\rho(X^k) = X *_\hbar^K k$, $X \in \mathsf{Lin}(\mathfrak{g}^*)$, $k \geq 0$. It was shown that the ρ_r's in $\rho = I + \sum_{r \geq 1} \hbar^r \rho_r$ are differential operators. Here we have only to solve the recurrence relation for ρ_r appearing in the proof of Theorem 2:

$$-\delta\rho_r(X, X^{k-1}) = C_r^K(X, X^{k-1}) + \sum_{\substack{a+b=r \\ a,b \geq 1}} C_a^K(X, \rho_b(X^{k-1})), \tag{18}$$

where $X \in \mathsf{Lin}(\mathfrak{g}^*), k \geq 1, r \geq 1$. According to Lemma 1, there exist differential operators η_r such that $C_r^K(X, X^k) = \delta\eta_r(X, X^k)$. From Lemma 7 it follows that

$$\eta_r = -2^r(r-1)!w(\Gamma_1^{(r)}) \sum_{1 \leq i_* \leq d} \mathsf{Tr}(ad_{x^{i_1}} \cdots ad_{x^{i_r}})\, \partial_{i_1 \cdots i_r}, \quad r \geq 2.$$

For each $r \geq 2$, η_r is a differential operator with constant coefficients and is homogeneous of degree r in the derivatives. To a differential operator η on $\mathfrak{g}^*$ with constant coefficients we can associate a polynomial $\hat{\eta}$ on $\mathfrak{g} \sim \mathsf{Lin}(\mathfrak{g}^*)$. Here we have $\hat{\eta}_r(X) = -2^r(r-1)!w(\Gamma_1^{(r)})\mathsf{Tr}((ad_X)^r)$ and one can check that

$$\delta\eta_r(X, X^k) = -\frac{r}{k}\eta_r(X^k) = -\frac{r}{k}\,\frac{k!}{(k-r)!}\,\hat{\eta}_r(X)\,X^{k-r}. \tag{19}$$

The preceding implies that the ρ_r's, $r \geq 1$, have constant coefficients and are homogeneous of degree r. We have $\rho_1 = 0$ and a recurrence on r in Eq. (18) shows the property for all of the ρ_r's.

Using Eq. (19) we can express Eq. (18) in terms of the polynomials $\hat{\rho}_r$ and $\hat{\eta}_r$ and find that:

$$\hat{\rho}_r(X) = -\hat{\eta}_r(X) - \frac{1}{r} \sum_{\substack{a+b=r \\ a,b \geq 1}} a\hat{\eta}_a(X)\hat{\rho}_b(X).$$

By defining $\hat{\eta}_0(X)$ to be identically equal to zero and $\hat{\rho}_0(X)$ to be 1, we can rewrite the previous equation as

$$r\,\hat{\rho}_r(X) = -\sum_{\substack{a+b=r \\ a,b \geq 0}} a\hat{\eta}_a(X)\hat{\rho}_b(X), \tag{20}$$

then by considering the formal series $\hat{\rho}(X) \equiv I + \sum_{r \geq 1} \hbar^r \hat{\rho}_r(X)$ and $\hat{\eta}(X) \equiv \sum_{r \geq 2} \hbar^r \hat{\eta}_r(X)$ (recall that $\eta_1 = 0$), we see that Eq. (20) simply states that $\hat{\rho}'(X) = -\hat{\eta}'(X)\hat{\rho}(X)$ where the prime denotes formal derivative with respect to $\hbar$. Thus $\hat{\rho}(X) = \exp(-\hat{\eta}(X))$ and Eq. (17) follows.

For a nilpotent Lie algebra, all of the operators D_r do vanish. Hence we deduce the result of [1]:

Corollary 2 *For a nilpotent Lie algebra, Kontsevich star-product coincides with Gutt star-product.*

4.1 Remarks

The equivalence between Kontsevich and Gutt star-products shows us that, in the linear Poisson case, graphs with cycles play no role with respect to the associativity of the product. Here the contribution of these graphs is completely absorbed into the equivalence operator. In other words: Weights of the graphs

$\Gamma_1^{(r)}$ can be chosen arbitrarily and they do not affect the associativity of the star-product.

We suspect that the situation described above is the general one, i.e., for $\mathbb{R}^d$ endowed with any Poisson structure π, it would be possible to get a new star-product by removing graphs with cycles in Kontsevich's construction. We conjecture that the Weyl star-product associated with Kontsevich star-product $*_\hbar^K$ on $(\mathbb{R}^d, \pi)$ contains no cycle, and it is obtained from $*_\hbar^K$ by ignoring the graphs with cycles.

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